Dynamic model for high-speed rotors based on their experimental characterization

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Abstract: Rotating systems components such as rotors, have dynamic characteristics that are of great importance to understand because they may cause failure of turbomachinery. Therefore, it is required to study a dynamic model to predict some vibration characteristics, in this case, the natural frequencies and mode shapes (both of free vibration) of a centrifugal compressor shaft. The peculiarity of the dynamic model proposed is that using frequency and displacements values obtained experimentally, it is possible to calculate the mass and stiffness distribution of the shaft, and then use these values to estimate the theoretical modal parameters. The natural frequencies and mode shapes of the shaft were obtained with experimental modal analysis by using the impact test. The results predicted by the model are in good agreement with the experimental test. The model is also flexible with other geometries and has a great time and computing performance, which can be evaluated with respect to other commercial software in the future.

Keywords: Dynamic Systems, Eigenvalues, Eigenvectors, Vibrations, Transfer Matrix Method (TMM).

1. Introduction

The vibrations in turbomachinery are responsible for over 40% of the problems [1-4], where rotating parts are the most important source [5]. These problems affect principally the bearings supports because these are the physical support of the rotor system, which play an important role in the vibration modes and critical frequencies [6]. Therefore, many methods have been developed to predict failures by analyzing the dynamic response of rotating systems.

Y. Zhang and Z. Du [7] obtained the natural frequencies of a gas turbine rotor experimentally and found that each order natural frequency increases with contact stresses. They showed that using these frequencies and the finite element analysis software SAMCEF Field, the effect of the contact stiffness can be obtained. The natural frequencies of a rotor vary with many factors such as the structure, the supports and the accuracy of the model, so it becomes complex to get a good prediction [8]. Xu et al. [8] determined the natural frequencies and mode shapes of a rotor using finite element models in ANSYS considering the bearings stiffness and damping. They noted that the mode shapes define where the mass unbalance forces have to be positioned, and that using different bearings, the natural frequencies and critical speeds can be controlled. Lu et al. [9] indicate that each order of natural frequency of a gas turbine rotor increases with rod preloads, because the rigidity changes. These preloads have a significant effect on the deflections and stresses, which modifies the modal analysis. Taplak and Parlak [10] made a dynamical analysis of a gas turbine rotor using the Dynrot
program with code based on the finite element method, they found the system has an unstable behavior when it is near to the critical speeds, and that small imbalance values do not affect the behavior but decrease the critical speeds. Pagar and Gawande [11] obtained the natural frequencies and mode shapes of a rotor shaft using ANSYS, and validated the results with an experimental test using FFT (Fast Fourier Transform) analyzer. If the system is modified with mountings and accessories, then stiffness increases and therefore the natural frequency. The results were in good agreement, although there was no feedback to the model with experimental data. Wu [12] presented a scale-down model to predict the free or forced lateral vibrations of a rotor-bearing system and showed that scaling laws and scaling factors approach the prediction of the full-size rotor from its scale model. Fegade et al. [13] presented a harmonic analysis to identify frequency of a system with different configuration of bearings using ANSYS. They used the unbalance of the rotor as excitation to perform the analysis, and mentioned that rotating critical speeds are associated with high vibration amplitude. In this case, a system with symmetric orthotropic bearings gave the less critical speed, so this kind of analysis is important to know the vibration amplitude response for minimizing the noise of the rotor. Ramirez et al. [14] used a finite element model of a Jeffcott rotor in ANSYS, approaching the stiffness and damping coefficients of hydrodynamic bearings. It was performed a harmonic analysis to determine the vibrational response in steady and transient state. The transient analysis showed the required excitation of the rotor to go through a natural frequency. The modal analysis reveals if any natural frequencies will be near the operating speed, and it is necessary to carry out a harmonic analysis. The natural mode shapes help to identify if the motion is near the bearing supports, which could affect the rotor critical speeds [15]. The natural frequencies of cantilever beam using FEM were estimated by Behera et al. [16], where errors up to 4% in the first three natural frequencies in relation to experimental results where shown. The natural frequencies of a rotor with 3 disks with different diameter, thickness and a uniform shaft where calculated by Entezari et al. [17], where errors up to 12.36% are shown, using various models in 1D FE with Carrera Unified Formulation in relation with 3D FE in ANSYS. Bakhtiari-Nejad et al. [18] used an analytical estimate based on the Rayleigh’s method to calculate the natural frequencies and modal shapes of a beam with different cracks configurations. Relative errors up to 66% with respect to an exact numerical method were shown, taking into account the first three natural frequencies and the deepest cracks.

The Transfer Matrix Method (TMM) has been applied to the solution of various engineering problems such as free and forced vibration of engineering structures. The state variables for FEM are displacements, velocities and accelerations, but for the TMM are displacements, slopes, bending moments and shearing forces. Therefore, the TMM would be more effective than FEM [19]. A dynamic model for free vibration of arch bridges was proposed by Kang et al. [20] based on the TMM to calculate the natural frequencies and modal shapes. Errors up to 4.2% are shown, with respect to the FEM by analyzing the first seven natural frequencies and different boundary conditions. Al-Bahkali and ElMadany [21] used the TMM to develop a graphical interface in MATLAB, to obtain the critical speed and the response to the imbalance of rotating machinery. The estimation by the TMM and the experiments are close enough for a rotor with two disks and two simple supports. Ellakany [22] developed a numerical model to calculate higher natural frequencies of composite beams based on Riccati TMM. The Riccati TMM gets better accuracy for higher order of natural than the conventional TMM, where the conventional TMM showed errors up to 26.8% until the 14th frequency. Zu and Ji [23] proposed an improved TMM for rotor-bearing systems with nonlinear bearings, where demonstrate the influence of nonlinear bearings on the dynamic response of the system. Tamrakar and Mittal [24] showed that the shear effect and rotary inertia improve the prediction of the dynamic performance of a rotor system. It is shown that when using different theories (Euler-Bernoulli, Rayleigh, and Timoshenko) to model the beam element, errors up to 10.23% in the natural frequencies are observed with respect to the experimental measurements.
There are still considerable relative errors between vibration parameters estimated by theoretical and experimental models. In this work, a method that works as a kind of theoretical-experimental hybrid is proposed, that allows calculating theoretical natural frequencies of a shaft from taking the experimental natural frequencies and mode shapes, without deep knowledge of the geometry. The natural frequencies by the TMM were estimated in order to compare the proposed model results with a well-founded numerical method.

2. Materials and Methods

2.1. Experimental Method

A turbocompressor shaft of 33.36 kg was used, Figure 1. The material properties of the shaft are shown in Table 1. The shaft was hanged by closest points to the nodes of the majority of the mode shapes predicted by the experimental model. Thereafter, it was impacted with an instrumented hammer (model PCB 086C03) to measure the response of the acceleration using two miniature triaxial piezoelectric CCLD accelerometers (model Brüel & Kjaer type 4506). A total of 6 natural frequencies were detected, so the accelerometer was moved on a total of 6 points along the shaft to get the mode shapes.

Table 1. Material properties.

<table>
<thead>
<tr>
<th>Steel AISI 4340</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>205 GPa</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.29</td>
</tr>
<tr>
<td>$\rho$</td>
<td>7850 kg/m$^3$</td>
</tr>
</tbody>
</table>

$^1$ Modulus of elasticity, $^2$ Poisson’s ratio.

When the shaft is excited, the transfer function is obtained and is processed with an acquisition and processing code developed for this purpose, which makes it easy to adjust the parameters as sample frequency, record length, profits, etc., as well as the calculation of transfer functions and averages. Only the first three normalized mode shapes obtained experimentally are shown in Figures 2-4.
2.2. Dynamic Model Approach

The equation of motion for the complete system is represented with the following model

\[ [M]\ddot{X} + [C]\dot{X} + [K]X = \{U(t)\}, \]  

(1)

where, M, C and K are the mass, damping and stiffness matrix, respectively. As the particular interest of this research is the free vibration, the equation is given by

\[ [M]\ddot{X} + [K]X = 0, \]  

(2)

The rotor shaft showed six experimental natural frequencies, which means that six degrees of freedom are required. The natural frequencies obtained experimentally, for lateral vibration, gives the number of the lumped dynamic equations. The system equations can be represented as follows:

\[
\begin{align*}
    u_1(t) &= m_1\ddot{x}_1 + k_2(x_1 - x_2) + k_1x_1 \\
    u_2(t) &= m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3(x_2 - x_3) \\
    u_3(t) &= m_3\ddot{x}_3 + k_3(x_3 - x_2) + k_4(x_3 - x_4) \\
    u_4(t) &= m_4\ddot{x}_4 + k_4(x_4 - x_3) + k_5(x_4 - x_5) \\
    u_5(t) &= m_5\ddot{x}_5 + k_5(x_5 - x_4) + k_6(x_5 - x_6) \\
    u_6(t) &= m_6\ddot{x}_6 + k_6(x_6 - x_5) + k_7x_6,
\end{align*}
\]  

(3)

Then converting to frequency domain and ordering in matrix form

\[ [U]_{36 \times 1} = [A]_{36 \times 36}[X]_{36 \times 1}, \]  

(4)

where A consists as follows

---

**Figure 3.** Second mode shape.

**Figure 4.** Third mode shape.
The matrix \( U \) is the force vector, which is considered unitary, to try to simulate the impacts of the hammer as unitary impulses. The matrix \( X \) contains all the experimental modal shapes in 6-displacement sections. The matrix \( A \) is a dynamic stiffness matrix, and it consists of the values of mass, stiffness and system frequency, where the mass and stiffness values are unknown. As the experimental frequencies and displacements are known, one can express the overall model in the following way

\[
[MK]_{13 \times 1} = [D]_{13 \times 36}^{-1}[U]_{36 \times 1},
\]

where \( D \) is a 36x13 matrix. Every six rows, the system shown below in equation 7, is repeated but changing the frequency and the corresponding displacements to the next modal shape.

\[
D = 
\begin{bmatrix}
1 \omega_0 - \omega_0 & 0 & 0 & 0 & 0 & -i \omega_0 \omega_0 & 0 & 0 & 0 & 0 \\
0 & 1 \omega_0 - \omega_0 & 0 & 0 & 0 & 0 & -i \omega_0 \omega_0 & 0 & 0 & 0 \\
0 & 0 & 1 \omega_0 - \omega_0 & 0 & 0 & 0 & 0 & -i \omega_0 \omega_0 & 0 & 0 \\
0 & 0 & 0 & 1 \omega_0 - \omega_0 & 0 & 0 & 0 & 0 & -i \omega_0 \omega_0 & 0 \\
0 & 0 & 0 & 0 & 1 \omega_0 - \omega_0 & 0 & 0 & 0 & 0 & -i \omega_0 \omega_0 \\
0 & 0 & 0 & 0 & 0 & 1 \omega_0 - \omega_0 & 0 & 0 & 0 & -i \omega_0 \omega_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \omega_0 - \omega_0 & 0 & 0 & -i \omega_0 \omega_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \omega_0 - \omega_0 & 0 & -i \omega_0 \omega_0 \\
\end{bmatrix}
\]

The resulted \( MK \) matrix is mass normalized to give the distribution of the stiffnesses and masses of the rotor. Then, all the stiffness and masses calculated are substituted in the first 6x6 dynamic stiffness matrix of the model in equation 5, but leaving \( \omega \) as the unknown variable. Therefore, the eigenvalues and consequently the natural frequencies are calculated from this dynamic stiffness matrix.

2.3. Transfer Matrix Method

The method consists of modeling a flexible rotor by a number of lumped inertias connected by massless elastic shaft sections [25, 26]. This method involves a small resulting system of equations and does not requires a deep knowledge of geometry. The point and field matrices are obtained from the equations of motion for the inertia and shaft elements, respectively. The Figure 5 shows free body diagrams of the inertia (disk) and shaft (elastic) elements, constituting the \( i \)th station. The variables of interest at station \( i \) are the displacement \( R_i \), slope \( \theta_i \), moment \( M_i \) and shear \( V_i \), where the superscripts \( r \) and \( l \) denote the right and left hand of the station.
Figure 5. Transfer-matrix state variables for \( i \)th rotor station.

The bracketed matrix of coefficients, in equation 8, is called the point matrix, and represents the inertia transfer matrix of the \( i \)th station.

\[
\begin{bmatrix}
R'_i \\
\theta'_i \\
M'_i \\
V'_i
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -I_{di} \omega^2 & 1 & 0 \\
-m_i \omega^2 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
R'_i \\
\theta'_i \\
M'_i \\
V'_i
\end{bmatrix}
\]  

(8)

where \( I_{di} \) is the mass moment of inertia of the symmetric disk about \( x \) or \( y \), and \( m_i \) is the mass of the disk.

The equation 8 can be stated more compactly as

\[
(S'_i) = [T_{si}] (S')_i
\]  

(9)

where \( (S')_i \) is identified as the state vector.

Statics and beam deflection theory give the transfer equations for the flexible shaft element in Figure 5. The bracketed matrix, in equation 10, is called the field matrix and relates the variables on the left and right hand sides of the \( i \)th field (beam).

\[
\begin{bmatrix}
R'_{i+1} \\
\theta'_{i+1} \\
M'_{i+1} \\
V'_{i+1}
\end{bmatrix} =
\begin{bmatrix}
1 & l_i & l_i^2 / 2EI_i & -l_i^3 / 6EI_i \\
0 & 1 & l_i / EI_i & -l_i^2 / 2EI_i \\
0 & 0 & 1 & -l_i \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
R'_i \\
\theta'_i \\
M'_i \\
V'_i
\end{bmatrix}
\]  

(10)

where \( l_i \) is the shaft station length, \( E \) is modulus of elasticity, and \( I_i \) is the area moment of inertia for the shaft station.

The equation 10 can be stated more compactly as

\[
(S'_i)_{i+1} = [T_{fi}] (S')_i
\]  

(11)

and combined with equation 9 to yield the overall transfer matrix

\[
(S'_i)_{i+1} = [T_{fi}] [T_{si}] (S')_i = [T_f] (S')_i
\]  

(12)

relating the variables on the left hand sides of station \( i \) and station \( i + 1 \).

The matrices are multiplied together using the transfer matrix definition of equation 12 to relate the boundary conditions at the left hand of station 1 to the boundary conditions at the left hand of last station \( n \), as indicated below
\[(S)_{n}^{l} = [T_{n-1}][T_{n-2}] \cdots [T_{1}](S)_{1}^{l}, \]  

(13)

The left and right hand side variables for last station \(n\) are related by the \(n\)th station transfer matrix

\[(S)_{n}^{r} = [T_{sn}](S)_{n}^{l}, \]  

(14)

Thus, the system transfer matrix from the left hand side of station 1 to right hand side of station \(n\) is

\[(S)_{n}^{r} = [T_{s1}][T_{s2}] \cdots [T_{s1}](S)_{1}^{r} = [T](S)_{1}^{r}, \]  

(15)

The matrix \([T]\) is a function of frequency \(\omega\). Substitution of the initial conditions of free vibration \(M_{n}^{r} = V_{n}^{r} = 0, M_{1}^{l} = V_{1}^{l} = 0\), gives

\[
\begin{bmatrix}
    T_{31} & T_{32} \\
    T_{41} & T_{42}
\end{bmatrix}
\begin{bmatrix}
    T_{41}
\end{bmatrix}
\begin{bmatrix}
    T_{31}T_{42} - T_{32}T_{41} = D(\omega) = 0
\end{bmatrix}
\]  

(16)

The equation 16 is a submatrix of \([T]\). The value of \(\omega\) which causes equation 16 to be satisfied is a natural frequency for the rotor in free-free condition. The lumped mass approximation of the shaft consisted of dividing the inertia of each section into equal parts, and lumped into the disks at each end of the section. The algorithm was programmed in MATLAB.

3. Results

Table 2 shows a summary comparison of the natural frequencies of the rotor shaft corresponding to the six modes of vibration measured experimentally, those predicted by the proposed method in this work for the lateral vibration modes and those predicted by TMM. It can be observed that the estimation of the theoretical model approximates the experimental, which showed an average percentage of relative error 18.87\%. It can be said that the proposed dynamic model results are in good agreement with the experiments.

<table>
<thead>
<tr>
<th>No.</th>
<th>Experimental (Hz)</th>
<th>Proposed Method (Hz)</th>
<th>TMM (Hz)</th>
<th>Error Proposed Method (%)</th>
<th>Error TMM (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>398.73</td>
<td>262.44</td>
<td>389.84</td>
<td>34.17</td>
<td>2.22</td>
</tr>
<tr>
<td>2</td>
<td>981.69</td>
<td>894.13</td>
<td>949.62</td>
<td>8.91</td>
<td>3.26</td>
</tr>
<tr>
<td>3</td>
<td>1614.8</td>
<td>1693.56</td>
<td>1582.98</td>
<td>4.87</td>
<td>1.97</td>
</tr>
<tr>
<td>4</td>
<td>2289.8</td>
<td>1770.75</td>
<td>2232.87</td>
<td>22.66</td>
<td>2.48</td>
</tr>
<tr>
<td>5</td>
<td>3222.6</td>
<td>2819.74</td>
<td>2927.48</td>
<td>12.5</td>
<td>9.15</td>
</tr>
<tr>
<td>6</td>
<td>4463.8</td>
<td>3121.81</td>
<td>4152.94</td>
<td>30.1</td>
<td>6.96</td>
</tr>
</tbody>
</table>

4. Discussion

Mogeiner et al. [27] calculated the modal parameters of a rotor using a FE branched model. The predicted lateral natural frequencies showed an error up to 6.5\% compared with the measured natural frequencies. Jalali et al. [28] determined the natural frequencies of a rotor using a 1D-beam model. The natural frequencies showed errors up to 22\% in relation with 3D FEM and up to 13\% with experimental results. Boiangiu et al. [29] used the TMM for a vibration analysis of a conic beam in cantilever. Considering cylindrical elements, the percentage error in the first five natural frequencies
varied between 2.2 to 7.3% compared with experimental results. Bencomo et al. [30] calculated the natural frequencies of an experimental flexible rotor with a modified TMM. A percentage error up to 6.62% in relation with experimental modal analysis, and up to 12.41% compared to run-down measures was reported. The natural frequencies and mode shapes of flexible rotors was calculated with the TMM by Violante et al. [31]. An error up to 14.4% in natural frequencies compared with an experimental test rig, and up to 4.9% with a real gas turbine rotor was reported. A numerical assembly method (NAM) to investigate the free vibration of a Timoshenko beam with multiple point masses and different kind of springs was proposed by Lin [32]. Errors up to 82% in the natural frequencies with respect to an Euler-Bernoulli beam and different configurations of point masses, inertias and springs, were shown.

It is to emphasize that the estimated frequencies correspond to the primary or crude model, without using recursive algorithms, so greater differences could be reduced by subsequent adjustments. There are several ways to determine the modal frequencies; however, most authors use the TMM and FEM, where the latter requires more computing performance and have the disadvantage of not be able to feedback control systems in operation.

5. Conclusions

This first model approximation showed good agreement with experimental results and is validated because is based on experimental input data. The model is also flexible with other geometries and has a great time and computing performance, which can be evaluated with respect to other commercial software in the future. It is intended to estimate the mode shapes with improving this model in future works. Depth knowledge in rotordynamics can contribute to the development of more reliable and predictable systems and support design tasks, monitoring and maintenance.

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Conflicts of Interest: “The authors declare no conflict of interest.”

References


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